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INVESTIGATION OF THE STABILITY OF THE VERTICAL POSITION OF A GYROSCOPE OF VARIABLE MASS

[Following is a translation of an article written by V. S. Novoselov in Vestnik Leningradskogo Universiteta (Herald of Leningrad University, No. 19, Ser Mat, Mekh i Astron, No. 4, 1959, pages 121-129.]

This article is devoted to a purely theoretical study of the stability of the vertical position of a gyroscope of a variable mass, the particles of which may possess an internal motion. Technical features are not considered. Basic results of this article were presented in the author's thesis [1].

1

1. Let us obtain the "apex" equation [2] of a gyroscope of variable mass. By this term we understand a body of variable mass that has kinetic symmetry and a stationary point. We shall also assume that a process of ejection but no accretion of particles might be taking place. The directions of the principal axes of the body are assumed to be fixed.

The formula of the variation of the kinetic moment of a body of a variable mass with respect to a stationary point has the form:

$$\frac{D}{Dt}\bar{I} = \bar{L} + \bar{M} \qquad (1,1)$$

for the general case, allowing for the internal movement of particles [3].

In this formula $\bar{I} = Ap\bar{i} + Aq\bar{j} + Cr\bar{k}$ is the aforementioned kinetic moment expressed in the principal axes with the orthonomeal vectors $\bar{i}$, $\bar{j}$, $\bar{k}$; A and C are the moments of inertia; p, q, r the projections of the angular velocity. Further: D/Dt is the symbol of the derivative with respect to time, which is computed on the assumption that all point masses of the rigid body are fixed. $\bar{L}$ and $\bar{M}$ are the moments of the acting forces and of forces resulting from the variability of the total mass.

Let us use $\bar{v}$ for the velocity of the end-point of the vector $\bar{k}$ which corresponds to the gyroscope's apex. We find:

$$p\bar{i} + q\bar{j} = \bar{k} \times \bar{v}. \tag{1,2}$$

Consequently, the kinetic moment shall be written in the form:

$$\bar{I} = A(\bar{k} \times \bar{v}) + Cr\bar{k}. \tag{1,3}$$

On the basis of the formula (1.3) the equation (1,1) assumes the form:

$$A(\bar{k} \times \dot{\bar{v}}) + C\dot{r}\bar{k} + Cr\bar{v} = \bar{L} + \bar{M}. \tag{1,4}$$

The moments of the active forces shall be composed of the moment of the force of gravity and the moments of the resistances to the gyroscope's motion, which will include the damping forces of the ejected particles. On neglecting friction in the suspension and using formula (1,2) we may assume:

$$\bar{L} = h\bar{k} \times m\bar{g} - v_1\bar{k} \times \bar{v} - v_2 r\bar{k}. \tag{1,5}$$

Here h is the distance of the gyroscope's center of gravity from the point of suspension. The quantity h will be greater than zero, equal to zero, or smaller than zero, depending on whether the center of gravity is above the stationary point, coincides with it, or is below it. v_1 is a positive function of time, and v_2 a positive constant. (The quantities v_1 and v_2 stand for resistance and damping). For a rapid rotation the resistance $v_2 r$ must be replaced by the experimentally determined function f(r) of the resistance.

It is not too difficult to realize a construction of a gyroscope, for which the relative kinetic moment of particles moving in the gyroscope will be equal to zero. In the case of such a construction the moment M [3] would be written in the form

$$\bar{M} = -\bar{I}^r - \bar{I}^e - \frac{d\bar{I}}{dt} + \frac{D\bar{I}}{Dt}, \tag{1,6}$$

Here $\bar{I}^r$ is the loss per second of the relative kinetic moment of the particles across the gyroscope's surface; $\bar{I}^e$ is the loss per second of the transferable kinetic moment of the particles through the same surface.

Assuming the process of the ejection of particles to be symmetric we shall have:

$$\bar{I}^r = -K\bar{k},$$

where K is a known function of time. And we shall have further:

$$\frac{d\bar{I}}{dt} - \frac{D\bar{I}}{Dt} = \dot{A}(\bar{k}\times\bar{v}) + \dot{C}r\bar{k}; \qquad (1,7)$$

$$\bar{I}^e = \mu_1(\bar{k}\times\bar{v}) + \mu_2 r\bar{k}. \qquad (1,8)$$

The symbols μ_1, and μ_2 in the formula (1,8) are the known positive functions of time, corresponding to the moments of inertia of loss of mass per second. We shall consider gyroscopes for which $\mu_1 \geqslant |\dot{A}|$, and $\mu_2 \geqslant |\dot{C}|$. The functions of time K, μ_1 and μ_2 as well as A, C, h, m and v_1, can be determined experimentally upon the stand.

Taking the scalar and vector products of $\bar{k}$ and the equation (1,4), and taking into consideration formulas (1,5) -- (1,8), we obtain

$$C\dot{r} + (\mu_2 + \nu_2 + \dot{C})r = K; \qquad (1,9)$$

$$A\dot{\bar{v}} = Cr(\bar{k}\times\bar{v}) - Av^2\bar{k} + h\bar{k}\times(\bar{k}\times m\bar{g}) - (\mu_1 + \nu_1 + \dot{A})\bar{v}. \qquad (1,10)$$

The equation (1,9) shall be called "equation of the apex" as used by Yu. A. Krutkov.

2. Equation (1,9) has the Solution

$$r = e^{-\int_0^t \frac{\mu_2+\nu_2+\dot{C}}{C}dt}\left(r_0 + \int_0^t \frac{K}{C} e^{\int_0^t \frac{\mu_2+\nu_2+\dot{C}}{C}dt}\right). \qquad (1,11)$$

For a very rapid rotation of the gyroscope, when the resistance may no longer be considered linear, the equation of rotation about the instrument's own axis can be integrated approximately.

It is obvious that equation (1,10) has a particular solution

$$\bar{k} = \bar{a}, \ \bar{v} = 0. \qquad (1,12)$$

where a is the unit vector of the vertical axis.

We shall study the stability of the vertical position (1,12). For this we shall have equations by the first approximation.

Let k = a + $\bar{R}$. We will consider the fixed system of the coordinates (x, y, z), with origin o at the end-point of the vector $\bar{a}$. The axes ox, oy shall be horizontal, while oz is directed vertically downwards.

We shall have the formulas

$$\bar{v} = \dot{\bar{R}}; \ \dot{\bar{v}} = \ddot{\bar{R}};$$

$$h\bar{k} \times (\bar{k} \times m\bar{g}) = mgh\,|\bar{u}(R^2 + \bar{a}\bar{R}) - \bar{R}(1 + \bar{u}\bar{R})|. \qquad (1,10)$$

[1,10]

On projecting the equation (1,10) onto the axes ox and oy by the first approximation with respect to quantities x and z and their derivatives, we obtain:

$$\left.\begin{aligned} \ddot{x} &= -\gamma\dot{x} - \beta\dot{y} + ax, \\ \ddot{y} &= \beta\dot{x} - \gamma\dot{y} + ay, \end{aligned}\right\} \qquad (1,13)$$

where $a = \frac{mgh}{A}$, $\beta = \frac{Cr}{A}$, $\gamma = \mu_1 + \nu_1 + \dot{A}$ are some bounded functions of time.

For the projection z we have the formula

$$z = 1 - \sqrt{1 - x^2 - y^2},$$

Therefore, it follows from stability with respect to x and y and their derivatives, and of their derivative functions, that motion in the z direction is also stable.

2

Multiplying the first equation of the system (1,13) by $\dot{x}$ and the second by $\dot{y}$ we obtain, on adding,

$$\frac{d}{dt}V = -2\gamma(\dot{x}^2 + \dot{y}^2) - \dot{a}(x^2 + y^2), \qquad (2.1)$$

where we introduce the notation

$$V = \dot{x}^2 + \dot{y}^2 - a(x^2 + y^2). \qquad (2,2)$$

Let the center of gravity be located below the stationary point; then $h < 0$ and consequently $\alpha < 0$. The function V shall be definite positive, since $\mu_1 \geq |\dot{A}|$ and $\gamma > 0$. Let α now be a monotonically increasing function. In that case the derivative $\frac{d}{dt}V$ is definite negative, as follows from formula (2,1). For sufficiently small initial conditions, this derivative will remain negative definite if computed using the non-linear equations of the gyroscopic motion, because terms of the second and of higher orders cannot affect the sign in the right member of the relation (2,1). Hence, by the fundamental theorem of A. M. Lyapunov's second method [4] we conclude that the vertical position of the gyroscope of variable mass is asymptotically stable in the case considered.

Let us suppose that the center of gravity coincides with the stationary point, i.e. $\alpha = 0$. Using the same theorem of Lyapunov we shall prove asymptotic stability with respect to $\dot{x}$ and $\dot{y}$. Since these latter are small for small initial conditions, we can use for their evaluation the equations ofthe first approximation (1,13), which in this case admit the integral

$$\dot{x}^2 + \dot{y}^2 = (\dot{x}_0^2 + \dot{y}_0^2) e^{-2\int_0^t \gamma dt} \tag{2,3}$$

Denoting by the symbol "*" the averaging we find for the case when $\alpha = 0$ from the equations (1,13)

$$\dot{x} - \dot{x}_0 = -\gamma^* (x - x_0) - \beta^* (y - y_0),$$
$$\dot{y} - \dot{y}_0 = \beta^* (x - x_0) - \gamma^* (y - y_0).$$

Therefore, for small velocities, the coordinates will suffer negligibly from the initial values.

2. If the process of variation in mass occurs only in a finite time interval $[0, T]$ and should the resistance of the air to the rotation of the gyroscope about its own axis be negligible, the coefficients factors of the system (1,13) will be constant for t $\geq$ T.

The characteristic equation shall be written as:

$$\lambda^4 + 2\gamma\lambda^3 + (\gamma^2 - 2a + \beta^2)\lambda^2 - 2a\gamma\lambda + a^2 = 0.$$

In order to satisfy Hurwitz's conditions it is both sufficient and necessary that in our case $\alpha < 0$. Therefore, if at the instant t the quantity h is negative, the vertical position of the gyroscope shall be stable. The nature of the gyroscope's motion can be examined by the method of B. V. Bulgakov [5].

Let the coefficients in the equations of the system (1,13) approach finite limits as t → infinity. Assuming $v_2 = 0$, we obtain a boundary system with constant coefficients. If the limit is less than zero, then by the theorem of K. P. Persidskiy [6] the vertical position of the gyroscope is also stable in this case.

3

1. Should internal motion of the particles in the gyroscope be absent, as for example in the case of

$$\mu_1 = -\dot{A};\ \mu_2 = -\dot{C}.$$

we shall have a purely superficial burning away.

We shall neglect the air resistance. Then as a consequence of formulas (1,11) and (1,13) we shall find:

$$r = r_0 + \int_0^t \frac{K}{C}\,dt, \tag{3.1}$$

$$\left.\begin{aligned} \dot{x} &= -\beta y + ax, \\ \dot{y} &= \beta x + ay. \end{aligned}\right\} \tag{3.2}$$

2. As a preliminary we will prove a theorem referring to the approximate solution of a system of linear differential equations with variable coefficients.

Let us suppose we have a linear system:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j \quad (i = 1, 2, \ldots, n), \tag{3.3}$$

where a_{ij} are continuous functions of time, and where $|a_{ij}| < a$ for $0 \leq t <$ infinity. We shall consider an arbitrary time interval [0, T] and subdivide it by points $t_0, t_1, t_2, \ldots, t_{m-1} = T - \chi, t_m = T + \tau - \chi$ into m parts so that $t_{i+1} - t_i = \tau$, $\chi \leq \tau$. We write the approximating system

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a'_{ij} y_j. \tag{3.4}$$

with the notation:

$$a'_{ij} = \begin{cases} a_{ij}(0) & \text{при } 0 \leqslant t < t_1, \\ a_{ij}(t_1) & \text{при } t_1 \leqslant t < t_2, \\ \vdots & \vdots \qquad \vdots \\ a_{ij}(t_{m-1}) & \text{при } t_{m-1} \leqslant t < T. \end{cases} \tag{3.5}$$

The theorem consists in the following: for any arbitrarily small $\varepsilon > 0$ and an arbitrarily large $T > 0$ we can find a sufficiently small $\tau > 0$, so that for $x_{i0} = y_{i0}$ and $0 \leq t \leq T$ we shall have:

$$|x_i - y_i| < \varepsilon. \tag{3.6}$$

The continuous solution of the system (3,4) with the initial conditions y_{i0} will be unique. Now we shall use the method of successive approximations for computing y_i, taking x_i as the zero approximation. By virtue of the system (3,4) we have:

$$y_i = y_{i0} + \int_0^t \sum_{j=1}^{n} a'_{ij} y_j\,dt.$$

The first approximation shall be written in the form:

$$y_i^{(1)} = x_i + \int_0^t \sum_{j=1}^n (a_{ij}' - a_{ij}) x_j \, dt.$$

This gives the estimate

$$|y_i^{(1)} - x_i| < \delta A(T) nt,$$

in which we make use of the boundedness of the functions under the integral sign in the expressions

$$|a_{ij}' - a_{ij}| < \delta, \quad |x_j| < A(T).$$

Further, we have

$$y_i^{(2)} - y_i^{(1)} = \int_0^t \sum_j a_{ij}' (y_j^{(1)} - x_j) \, dt;$$

$$|y_i^{(2)} - y_i^{(1)}| < \delta \frac{A(T)}{a} \frac{(ant)^2}{2!}.$$

Consequently

$$|y_i^{(k)} - y_i^{(k-1)}| < \delta \frac{A(T)}{a} \frac{(ant)^k}{k!}.$$

Thus we arrive at the following result:

$$y_i = x_i + \sum_{n=1}^{\infty} (y_i^{(n)} - y_i^{(n-1)});$$

$$|y_i - x_i| < \delta \frac{A(T)}{a} (e^{anT} - 1).$$

The functions a_{ij} are continuous, therefore they shall be uniformly continuous in the closed interval [0, T]. Consequently, a value of τ can be found for which

$$\delta < \varepsilon \frac{a}{A(T)} e^{-anT}.$$

This gives the estimate (3,6). It must be mentioned that this estimate shall be uniform relative to T as τ varies.

3. Let us examine the stability of the zero solution of the system (3,2). This system is linear and the theorem just proven applies.

Let us consider the approximating system:

$$\ddot{x}_1 = -\beta'\dot{y}_1 + \alpha' x_1, \quad \ddot{y}_1 = \beta'\dot{x}_1 + \alpha' y_1, \tag{3,7}$$

where α' and β' are constants obtained by the approximation rule of (3,5). Setting $u = x_1 + iy_1$ ($i = \sqrt{-1}$), we reduce the system (3,7) to a single equation

$$\ddot{u} + i\beta'\dot{u} - \alpha' u = 0. \tag{3,8}$$

For the j^{th} interval we have

$$\ddot{u}_j + i\beta_j\dot{u} + a_j u_j = 0. \tag{3,9}$$

Let us set

$$u_j = w_j e^{i\frac{\beta_j}{2}\tau}, \tag{3,10}$$

which yields the equation

$$\ddot{w}_j + \left(\frac{\beta_j^2}{4} - a_j\right) w_j = 0. \tag{3,11}$$

for the variable w_j.

Formulas (3,10) and (3,11) show that the motion of a point having the coordinates x_1 and y_1 can be represented as a set of relative motion in accordance with equation (3,11) and translatory motion of uniform rotation with an angular velocity $\beta_j/2$.

We shall note that the stability of the sequence of functions w_j, $\dot{w}_j$ implies the stability of the zero solution of system (3,7), since by formula (3,10) we have

$$|u_j| = |w_j|, \quad |\dot{u}_j| = \left|\dot{w}_j + i\frac{\beta_j}{2} w_j\right|. \tag{3,12}$$

A solution in the variables u and $\dot{u}$ must be continuous, hence by virtue of (3,12) we have the following transition conditions from the interval j to the interval j + 1:

$$\rho_j(\tau) = \rho_{j+1}(0), \quad \dot{\rho}_j(\tau) = \dot{\rho}_{j+1}(0), \quad \omega_j(\tau) - \frac{\beta_j}{2} = \omega_{j+1}(0), \tag{3,13}$$

where ρ_j is the modulus of the complex number ω_j; ω_j is the angular velocity of relative motion or a derivative with respect to time whose argument is the complex number ω_j; and $B_j = \beta_{j+1} - \beta_j$.

Let us suppose that $\beta^2/4 - \alpha > 0$. Denoting by b_j^2 the quantity indicated on the j^{th} interval, we obtain the following obvious integrals of the equation (3,11):

$$v_0^2 + b_j^2 \rho_j^2 = H_j, \quad H_j = \text{const}, \tag{3.14}$$

$$\omega_j \rho_j^2 = \sigma_j, \quad \sigma_j = \text{const}, \tag{3.15}$$

where $v_j = \sqrt{\dot\rho_j^2 + \omega_j^2 \rho_j^2}$ is the relative velocity on the j^{th} interval. By virtue of formulas (3,13) - (3,15) the following relation will hold:

$$\sigma_j = \sigma_0 - \frac{1}{2} \sum_{i=0}^{j-1} B_i \rho_i(\tau). \tag{3.16}$$

Let us suppose b^2 is decreasing (not necessarily rapidly). Then by virtue of the formulas (3,13) - (3,16) we have:

$$H_{j+1} = H_0 + \sum_{i=0}^{j-1} \left(b_{i+1}^2 - b_i^2 + \frac{B_i}{2} \sum_{k=i+1}^{j} B_k \right) \rho_i^2(\tau) + \\ + (b_{j+1}^2 - b_j^2)\rho_j^2(\tau) - \sigma_0(\beta_{j+1} - \beta_0) + \frac{1}{4} \sum_{i=0}^{j} B_i^2 \rho_i^2(\tau). \tag{3,17}$$

The expression in square brackets in (3,17) equals:

$$\left(\frac{\beta_{i+1}}{2}\dot\beta_i - \dot\alpha_i\right)\tau + \left(\frac{d^2 b^2}{dt^2}\Big|_i^{*} + \ddot\beta\Big|_i^{*}\frac{\beta_{i+2} - \beta_{i+1}}{2} - \frac{\dot\beta_i}{2}\dot\beta\Big|_i\right)\tau^2. \tag{3.18}$$

In it the index "*" indicates the averaging in the indicated i^{th} interval α_i and β_i are derivatives computed at the instant

Since b^2 is decreasing we can expect that for sufficiently high relative velocities and a sufficient momentum K, $\frac{\beta_{min}}{2}\dot\beta - \dot\alpha < 0$. Consequently, for any value of the indices i and j we shall have

$$\frac{\beta_{i+1}}{2}\dot\beta_i - \dot\alpha_i \leq 0. \tag{3.19}$$

From the formulas (3,17) - (3,18) we obtain:

$$H_{i+1} \leq H_0 + 2|\sigma_0|\beta_{max} + MN(T)\tau(T+\tau),$$

in which we use the notations:

$$M = \sup\left(\left|\frac{d^2b^2}{dt^2}\right| + |\dot{\beta}||\beta| + \frac{3}{4}|\dot{\beta}|^2\right), \quad N(T) = \sup \beta^2.$$

Suppose the initial conditions are sufficiently small,

$$|v^0_{r}, \rho_0| < \delta,$$

then letting

$$\tau < \left\{\frac{\delta^2}{MN(T)\left(T+\frac{1}{2}\right)}, \frac{1}{2}\right\},$$

we obtain the inequality

$$v_r^2 + b^2\rho^2 < (2 + b_0^2 + 2\sup\beta)\delta^2 \qquad (3,20)$$

The estimate (3,20) is independent of T, consequently it is valid for the interval $0 \leq T <$ infinity.

The case where β and b^2 increase is also of interest. (The increase does not have to be fast). On account of the integral (3,14), the expression

$$\frac{v_{r_j}^2}{b_j^{2p+2}} + \frac{\rho_j^2}{b_j^{2p}} = G_j \quad p = \text{const}, \quad G_j = \text{const} \qquad (3,21)$$

will also be an integral of the equation (2,7) on the j^{th} time interval.

Due to the formulas (3,13) - (3,16) and (3,21), we have

$$G_j \leq G_0 - \varepsilon_0 \sum_{i=0}^{j} \frac{B_i}{b_{i+1}^{2p+2}} + \frac{1}{4}\sum_{i=0}^{j} B_i^2 \frac{\rho_i^2(\tau)}{b_{i+1}^{2p+2}} + \sum_{i=0}^{j-1} s_{ij}\rho_i^2(\tau),$$

where we introduce the notation

$$s_{ij} = \frac{1}{b_{i+1}^{2p}} - \frac{1}{b_i^{2p}} + \frac{B_i}{2}\left(\frac{B_{i+1}}{b_{i+2}^{2p+2}} + \frac{B_{i+2}}{b_{i+3}^{2p+2}} + \dots + \frac{B_j}{b_{j+1}^{2p+2}}\right)$$

Since b^2 is an increasing function, we have

$$s_{ij} \leq \frac{1}{b_{i+1}^{2p}} - \frac{1}{b_i^{2p}} + \frac{B_i}{2b_i^{2p+2}}(\beta_{j+1} - \beta_{i+1}).$$

Further on we will have

$$s_{ij} < \frac{1}{b_i^{2p+2}}\left|\frac{-(p+1)\varepsilon_0 + \sup\beta}{2}\dot{\beta}_i + p\dot{a}_i\right|\tau +$$

$$+\left[\frac{d^2}{dt^2}\frac{1}{b^{2p}}\left[_k + \frac{\ddot{\beta}\,l_i^{\bullet}}{2b_i^{2p+2}}(\beta_{l+1}-\beta_{i+1})\right]\tau^2.$$

Let us suppose that the relationship $\beta \leqslant q\beta_0$ is satisfied for a given number q. For sufficiently high relative velocities of the moving particles, β is considerably larger than $|\dot{\alpha}|$. Consequently, a sufficiently large number p can be found with the property that

$$-\frac{p-q+1}{2}\beta_0\dot{\beta}+p\dot{\alpha}<0.$$

From the above, for a sufficiently small τ and

$$\{v_r^0, \beta_0\}<\delta$$

the inequality

$$\frac{v_r^2}{b^{2p+2}}+\frac{\beta^2}{b^{2p}}<\left(1+\frac{1+2\sup\beta}{b_0^{2p+2}}+\frac{1}{b_0^{2p}}\right)\delta^2.$$

will be satisfied.

We have proved that in the two cases discussed the stability of the vertical position of a gyroscope of variable mass is valid by the first approximation. And we obtained, in addition, the estimates for the changes of the variables.

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